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DAN BRÂNZEI

FUNDAMENTALS OF SYNTHETIC PLANE GEOMETRY

PARALELA 45 EDITURA

IST CHAPTER
AN AXIOMATIC EDIFICATION OF GEOMETRY

§ 1. PRIMARY NOTIONS AND RELATIONS

We intend a familiarization with the axiomatic methods but not an axiomatic introduction to geometry. The intuitive geometry needs a continuous increase of abstract rigor. Some aspects, initially taken as obvious, sometimes appear later as uncertain and need justifications. The assertions **accepted as obvious** are the **axioms**. Only a few notions may be accepted with no definitions; it is said that these are **primary.** To polish the reasoning up to an exclusive use of the primary notions and of the axioms is seldom useful, often burdening, but it suggests some personal qualities for understanding the abstract.

For some didactic advantages we will present the Hilbert axiomatic system considering the theory of sets as known. In § 2 the primary notions: plane, point, straight-line, and a primary relation of incidence are introduced. A set π, called plane is given; its elements are called points and noted by Latin majuscules. A first step towards the *structure* of the set π, the concept of the "alignment" of points, is given by a primary notion: *straight line.* Here, a straight line will be thought as a subset a of π. The family of the straight-lines will be noted by Δ; obviously $\Delta \subset \mathcal{P}(\pi)$. We will impose in § 2 that this family Δ satisfies some conditions, the *axioms of incidence*; later, from these axioms we will logically deduce a lot of other properties.

In § 3 another primary notion, the *sense of a straight-line d*, will be given as a binary relation in the set *d*; some conditions imposed on (the set of) these binary relation will be the *order axioms*. Using the incidence and the order axioms we will **define** the concepts of *segment, be between, ray, half-plane and angle*. § 4 is devoted to a model where these axioms are verified. In § 5 the *axioms of congruence* will be imposed to the primary relations of *congruence* (of segments and of angles). The *axioms of continuity and parallelism* introduced in § 6 and § 7 need no other primary notions (or relations).

From this concise presentation we may imagine the complexity of Hilbert's axiomatic. It is not a flat construction: the access to the upper floors is allowed only after a carefully consolidation of the base and previous floors.

A tripper climbing a mountain necessarily steps on a lot of stones; he pays no attention to each of these stones but he enjoys the landscape from more and more advantageous positions. A second or a third trip will offer him new understandings and satisfactions. For those who intend a specialization in geology the stones may present some interest. Now, we will begin a journey in the castle of Geometry; we hope that the castle would be admired and not the stones it is made from.

§ 2. INCIDENCE AXIOMS

These axioms are related to the sets π, of points, and Δ, of straight-lines; we pointed out that $\Delta \subset \mathcal{P}(\pi)$. The interpretation of incidence as *to belong to* allows the use of some formulations from the set theory as: contain, include, intersection and so on. The intuitive view of geometry allows some traditional expressions like: a straight-line *passes through* a point, or *cuts* or *meets* another line. Numbering the axioms we use the first letter I of the word "incidence".

Axiom I_1. *For any two distinct points there exists **a unique straight-line** containing them.*

We will note by AB the straight-line passing through the points A and B, but we will confer some other meanings to the symbol AB when misunderstandings are not possible.

Axiom I_2. *Each straight-line has **at least two** distinct points.*

Axiom I_3. *There exist three* non-colinear[1] *points on* π (i.e. there is no straight line containing them).

Theorem 1. *There exist at least three distinct points.*

Theorem 2. *There exist at least three distinct straight-lines.*

Theorem 3. *Two straight-lines having in common two distinct points coincide.*

So, for the straight- lines d_1, d_2 the following two related positions are possible: **secant:** $\exists ! A, A \in d_1 \,\&\, A \in d_2$; **parallel:** $d_1 \parallel d_2$, that is: the lines **coincide** $(d_1 = d_2)$ or they have no common points $d_1 \cap d_2 = \emptyset$.

The incidence axioms do not assure the existence of distinct parallel straight-lines. As they say nothing about the "shape" of a straight-line, we will *traditionally draw* them, using the "ruler".

[1] Author apologize: he knows that the usual ortography is **collinear** but, as he cannot understand an ethymologic reason, he prefers this abridge.

EXERCISES

1. Let $\pi = \{A, B\}$. Count the choices of Δ such that: a) no supplementary conditions appear; b) exactly one of the axioms I_1, I_2, I_3 fail to be verified; c) all the axioms I_1, I_2, I_3 are verified.

2. Consider again exercise 1 for $\pi = \{A, B, C\}$.

3. Prove theorems 1, 2, 3 from § 2.

4. Let $\pi = \{A, B, C, D\}$ and Δ the family of the subsets of π each of them containing exactly two elements. Ascertain if the incidence axioms and the conclusions of the theorems are verified. Count the couples of distinct parallel straight-lines.

5. Let $\pi = \{0,1,2,3,4\}$ and $\Delta = \{\{0,1\},\{1,2\},\{2,3\},\{0,3\},\{0,2,4\},\{1,4,3\}\}$. Are the axioms I_1, I_2, I_3 verified?

6. Which one of the axioms I_1, I_2, I_3 fails to be verified in the following examples: a) $\pi = \{A, B, C\}$, $\Delta = \pi$; b) $\pi = \{A, B, C\}$, $\Delta = \{\{A, B\}, \{A, C\}\}$; c) $\pi = \{A, B, C\}$, $\Delta = \{\{A, B\}, \{C\}\}$; d) $\pi = \{A, B, C, D\}$, $\Delta = \{\{A, B, C\}, \{B, C, D\}\}$?

7*[1]. We know that π contains exactly five points and Δ is such that the incidence axioms are verified. Find the minimum a and the maximum b of the elements of Δ. Prove that for each natural number n such that $a < n < b$, there exists at least one Δ containing exactly n straight-lines.

8**. Generalize the previous problem for π containing m points.

[1] We mark by * and ** the problems appreciated as difficult.

§ 3. ORDER AXIOMS

The notions of (total) **order relation, opposed relations** and **last element** are defined. (Here, for an order relation R $aRb \Rightarrow a \neq b$.) Using the primary notion of sense the first three order axioms are given; numbering the axioms we use the initial letter O of the word "order".

Axiom O_1. *For any straight-line d, there exists a set* $\Omega(d) = \{O_d, O_d^*\}$ *where the* ***senses*** O_d *and* O_d^* *are opposite order relations.*

Axiom O_2. *For any straight-line d, none of its senses admits a last element.*

Definition 4.[1)] *We say that the point B is between the points A and C and we write* $A-B-C$ *or* $B \in |AC|$ *if:* a) $A \neq B$ *and* $B \neq C$; b) *there exists a straight-line d containing the points* A, B, C; c) *there exists a sense* $O_d \in \Omega(d)$ *such that* AO_dB *and* BO_dC. ($|AC|$ is a set of points called *segment*).

Some remarkable properties of the ternary relation *to be between*, ? - ? - ?, are given by theorems 2, 3, 4, 5 and 6.

Theorem 2. *If* $A-B-C$, *then the points* A, B, C *are distinct and co-linear and we also have* $C-B-A$.

Theorem 3. *For any triplet* $\{A, B, C\}$, *at most a point can be between the two others.*

Theorem 4. *For any distinct points* A, B, *there exists a point C such that* $A-B-C$.

Axiom O_3. *For any straight-line d, the set* $\pi - d$ *is the reunion of two disjoint non empty sets* H_1 *and* H_2, *such that the arbitrary points* A, B *belong to the distinct sets if:* $d \cap |AB| \neq \emptyset$.

The sets H_1 and H_2 are called **half-planes (delimited by** *d*). If the point A does not belong to d, it belongs to exactly one of the sets H_1, H_2, which will be noted (d, A).

If the points A, B, C do not belong to a straight-line, we call **triangle** ABC the set $\Delta ABC = \{A, B, C\} \cup |BC| \cup |CA| \cup |AB|$; the points A, B, C are its **vertexes** and the (open) segments $|AB|$, $|BC|$, $|CA|$ are its **sides**. We will refer as T to the set of triangles of the plane π.

Theorem 5. (Pasch). *Let ABC be a triangle and d be a straight-line containing none of its vertexes; if d cuts a side, then it will cut one more.*

Theorem 5'. *A straight-line containing none of the vertexes of a triangle ABC cuts two or none of the triangle's sides.*

Theorem 6. *For any distinct points* A, B: $|AB| \neq \emptyset$.

For a point A belonging to the straight-line d, *the subsets of d:* $O_d\langle A\rangle = \{B | AO_dB\}$ and $O_d^*\langle A\rangle = \{C | CO_dA\}$ are called **closed rays** (with the origin A). The sets $d \setminus O_d\langle A\rangle = O_d^*\langle A\rangle \setminus \{O\}$ and $d \setminus O_d^*\{A\} = O_d\langle A\rangle \setminus \{O\}$ are called **open rays** (with the origin A). We will denote by $|AB$ or A_B the open ray of origin A containing the point B. For each of the four considered rays (two closed and two open) we say that d is the *support straight-line*. A (closed or open) ray h of the straight-line d (with O as an origin) given, there exists a sense $\preccurlyeq$ in $\Omega(d)$ such that $A \in h \Leftrightarrow O \preccurlyeq A$; the restriction to h of the order $\preccurlyeq$ it is called *sense of the ray h*.

We call (proper) **angle** a set of two <u>closed</u> rays having the same origin but different support straight-lines. We will note the angle $\{O_x, O_y\} = \{O_A, O_B\}$ by $\sphericalangle xOy$ or $\sphericalangle AOB$; we will say that O is the *vertex* of the angle and Ox, Oy are its *sides*. We will refer by $\mathcal{U}$ to the set of proper angles of the plane π.

We call **interior of the angle** $\sphericalangle xOy$ and we note it by Int$\sphericalangle xOy$ the intersection of two half-planes: one delimited by OX and containing $(OY$ and the other one delimited by OY and containing $(OX$. "The ray $|AZ$ is interior to the angle XOY" means that $A = O$ and $(AZ \subset \text{int} \sphericalangle XOY$; the notations $|OX -\ |OZ - |OY$ or $O_X - O_Z - O_Y$ are used. (So, the betweenness relation is extended for the rays with the same origin).

[1)] Numberings are those from the source book.

If the angles xOy and yOz have no common interior points we say that they are **adjacent**; in this case, if the rays Ox, Oz have the same support straight-line, we say that the angles $\sphericalangle xOy, \sphericalangle yOz$ are **supplementary adjacent.** (Later on, we will call *supplementary angles* the ones that are congruent to supplementary adjacent angles).

Theorem 7. *The ray $|OX$ is interior to the angle AOB iff*[1] $|AB| \cap |OX \neq \emptyset$.

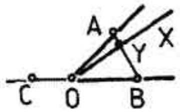

Fig. 1.3

For distinct parallel straight-lines d, d' we will say that the senses $\preccurlyeq \in \Omega(d)$, $\preccurlyeq' \in \Omega(d')$ **are coherent** if for any $A \prec B$ on d and $A' \neq B'$ on d' we have $A' \prec B'$ iff the points B, B' belong to the same half-plane delimited by AA'.

Theorem 8. *If d and d' are parallel straight-lines, then for any sense $\preccurlyeq$ in $\Omega(d)$ there exists exactly a coherent sense $\preccurlyeq'$ in $\Omega(d')$.*

EXERCISES

1. Prove theorem 1 from this paragraph.

2. Prove that the binary relation R and its opposed R^* are simultaneously (total) order relations.

3. Count all total orders for the sets with 2, 3 or 4 elements, concisely writing them down.

4. Give examples of orders with and without last elements.

5. Prove that a totally ordered set cannot have two last elements.

6. In the case when each element of Δ contains exactly two elements of π, prove that there exists a unique way to define the function Ω.

7. A set M is *dense* related to an order R if, for any couple (x, y) such that xRy (and $x \neq y$), there exists at least an element z such that xRz and zRy. Is a straight-line d dense related to the orders which correspond to its senses?

8*. A straight-line d and a point O on it are given. Let R be the binary relation on the set $d \setminus \{O\}$ defined by $XRY \Leftrightarrow O \notin |XY|$. Prove that R is an equivalence relation and point out its corresponding partition.

9*. To a straight-line d we associate a binary relation E_d on the set $\pi \setminus d$, defined by $XE_dY \Leftrightarrow |XY| \cap D \neq \emptyset$. Prove that E_d is an equivalence relation and point out its corresponding partition.

[1] Iff means: if and only if.

10. Let d be a straight-line, O one of its points and let A be a point not belonging to d. Prove that $|OA \subset (d, A)$.

11**. Given the betweenness relation on the straight-line d, construct a strict order relation R on d such that $X - Y - Z \Leftrightarrow [(XRY \,\&\, YRZ) \vee (ZRY \,\&\, YRX)]$.

§ 4*. A DISCRETE GEOMETRY

We image a flat, well-ordered plantation, endless in each direction, with very thin trees. Thinking to obtain such a plantation, we image that two perpendicular rays Ox, Oy were drawn. The trees are to be planted from meter to meter, on these straight-lines and on their equidistant parallels. To refer to a certain tree we need two integers x, y; first one represents its distance $|x|$ to Oy (to the right if $x > 0$, to the left if $x < 0$); the second one represents the distance $|y|$ to Ox (up if $y > 0$, down $y < 0$); if one of these two numbers vanishes, the tree should be on one of the lines Ox, Oy.

Now, considering the plantation as being a plane π, we try to define the set Δ of straight-lines. A straight-line d is a set $d_{a,b,c}$ "*of trees*": $(x, y) \in d_{a,b,c} \Leftrightarrow \Leftrightarrow ax + by + c = 0$. These subsets are given by integers a, b, c with no common divisor: (A method to solve the Diophantine equation is pointed out).

Theorem 1. *The mentioned system (π, Δ) verifies the axioms of incidence.*

We associate to the straight-line $d_{a,b,c}$ its senses: o_d and its opposite order o_d^* and we define $\Omega(d) = \{o_d, o_d^*\}$.

Theorem 2. *This function Ω verifies the order axioms.*

EXERCISES

1. Prove theorem 1.
2. Prove theorem 2.
3. For the above constructed model, say if the conclusion of the theorem 6, § 3 is true or not. Explain why.
4. Solve the Diophantine equations: a) $2x + 3y = 1$; b) $64x - 37y = 1$; c) $64x - 37y = 10$; d) $12345x - 6789y = 111$.